Full Length Research Paper

A New method for Solving Dual Fully Fuzzy Linear System

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Abstract. In this paper we intend to solve the dual fully fuzzy linear system of the form $\bar{A}_1 \otimes \bar{x} = \bar{A}_2 \otimes \bar{x} \oplus \bar{b}$ where $\bar{A}_1$ and $\bar{A}_2$ are n x n fuzzy matrices consisting of positive fuzzy numbers, the unknown vector $\bar{x}$ is a vector consisting of n positive fuzzy numbers and the constant $\bar{b}$ are vectors consisting of n positive fuzzy numbers, using QR decomposition method.

Key words: Dual fully fuzzy linear system, Triangular fuzzy numbers, QR Decomposition.

1. INTRODUCTION

In this paper we considered Dual fully fuzzy linear system of the form $\bar{A}_1 \otimes \bar{x} = \bar{A}_2 \otimes \bar{x} \oplus \bar{b}$, where $\bar{A}_1$ and $\bar{A}_2$ are n x n fuzzy matrices consisting of positive Triangular fuzzy numbers, the unknown vector $\bar{x}$ is a vector consisting of n positive Triangular fuzzy numbers and the constant $\bar{b}$ are vectors consisting of n positive Triangular fuzzy numbers. (Yosef Jafarzadeh 2011) propounds duality of fully fuzzy linear system of the form $\bar{A}_1 \otimes \bar{x} = \bar{b}$ using direct method. (Mosleh et al. 2011) suggested solution of fully fuzzy linear systems of the form $\bar{A}_1 \otimes \bar{x} = \bar{b}$ by ST method. (Neseri. S.H. et al. 2008) proposed a LU decomposition method for solving fully fuzzy linear system with triangular fuzzy numbers.

1.1 The structure of this paper is organized as follows

In Section 2, we present some basic concepts of fuzzy set theory and define a fully fuzzy linear system of equations. In Section 3, we have given the general model of dual fully fuzzy linear system. In Section 4, we extended QR decomposition method for solving a dual fully fuzzy linear system, In Section 5, deals with numerical examples for square and rectangular fuzzy matrices and verification of the results In Section 6 gives Conclusion and References.

2. PRELIMINARIES

Definition 2.1.

A fuzzy subset $\bar{A}$ of R is defined by its membership function $\mu_{\bar{A}}: R \rightarrow [0,1]$, which assigns a real number in the interval [0, 1], to each element $x \in R$, where the value of $\mu_{\bar{A}}$ at $x$ shows the grade of membership of $x$ in $\bar{A}$.

Definition 2.2.

A fuzzy number $\bar{A} = (m, \alpha, \beta)$ is said to be positive (negative), denoted by $\bar{A} > 0 (\bar{A} < 0)$, if its membership function $\mu_{\bar{A}}$ in the interval [0, 1], to each element $x \in R$, where the value of $\mu_{\bar{A}}$ at $x$ shows the grade of membership of $x$ in $\bar{A}$.

Definition 2.3.

A fuzzy number $\bar{A} = (m, \alpha, \beta)$ is said to be a triangular fuzzy number if its membership function is given by

$$\mu_{\bar{A}}(x) = \begin{cases} 1 - \frac{m-x}{\alpha}, & m - \alpha \leq x \leq m, \alpha > 0 \\ 1 - \frac{x-m}{\beta}, & m \leq x \leq n + \beta, \beta > 0 \\ 0, & \text{otherwise} \end{cases}$$

Definition 2.4.

A fuzzy number $\bar{A}$ is called positive (negative), denoted by $\bar{A} > 0 (\bar{A} < 0)$, if its membership function $\mu_{\bar{A}}(x)$ satisfies $\mu_{\bar{A}}(x) = 0, \forall x \leq 0 (\forall x \geq 0)$.

Definition 2.5.

Two fuzzy numbers $\bar{A} = (m, \alpha, \beta)$ and $\bar{B} = (n, \gamma, \delta)$ are said to be equal if and only if $m = n$, $\alpha = \gamma$ and $\beta = \delta$.
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Definition 2.5.

A Triangular fuzzy number \( \tilde{A} = (m, \alpha, \beta) \) is said to be zero triangular fuzzy number if and only if \( m = 0, \alpha = 0, \beta = 0 \).

Definition 2.6.

Let \( \tilde{A} = (a_{ij}) \) and \( \tilde{B} = (b_{ij}) \) be two \( m \times n \) and \( n \times p \) fuzzy matrices. We define \( \tilde{A} \otimes \tilde{B} = \tilde{C} = (c_{ij}) \) which is the \( m \times p \) matrix where

\[
\tilde{c}_{ij} = \sum_{k=1}^{n} a_{ik} \otimes b_{kj}
\]

Definition 2.7.

Arithmetic operations on triangular fuzzy numbers

Let \( \tilde{A}_1 = (m, \alpha, \beta) \) and \( \tilde{A}_2 = (n, \gamma, \delta) \) be two triangular fuzzy numbers then

(i) \( \tilde{A}_1 \oplus \tilde{A}_2 = (m+n, \alpha+\gamma, \beta+\delta) \)

(ii) \( \tilde{A}_1 = -(m, \alpha, \beta) = (-m, \alpha, \beta) \)

(iii) \( \tilde{A}_1 \geq 0 \) and \( \tilde{A}_2 \geq 0 \) then

\[
\tilde{A}_1 \otimes \tilde{A}_2 = (m, \alpha, \beta) \otimes (n, \gamma, \delta) = (mn, m\gamma + n\alpha, m\delta + n\beta)
\]

Definition 2.8.

A matrix \( \tilde{A} = (a_{ij}) \) is called a fuzzy matrix, if each element of \( \tilde{A} \) is a fuzzy number. A fuzzy matrix \( \tilde{A} \) will be positive and denoted by \( \tilde{A} > 0 \), if each element of \( \tilde{A} \) be positive. We may represent \( n \times n \) fuzzy matrix \( \tilde{A} = (a_{ij})_{n \times n} \), such that

\[
a_{ij} = (a_{ij}, m_{ij}, n_{ij}), \text{with the new notation} \quad \tilde{A} = (A, M, N) \text{ where} A = (a_{ij}), M = (m_{ij}), N = (n_{ij}) \text{ are three} \ n \times n \text{ crisp matrices.}
\]

Definition 2.9.

A square fuzzy matrix \( \tilde{A} = (a_{ij}) \) will be an upper triangular fuzzy matrix, if \( \tilde{a}_{ij} = \tilde{0} = (0,0,0) \) \( \forall \ i > j \), and a square fuzzy matrix \( \tilde{A} = (\tilde{a}_{ij}) \) will be a lower triangular fuzzy matrix, if \( \tilde{a}_{ij} = \tilde{0} = (0,0,0) \) \( \forall \ i < j \).

Definition 2.10.

Consider the \( n \times n \) fuzzy linear system of equations

\[
(\tilde{a}_{11} \otimes \tilde{x}_1) \oplus (\tilde{a}_{12} \otimes \tilde{x}_2) \oplus \ldots \oplus (\tilde{a}_{1n} \otimes \tilde{x}_n) = \tilde{b}_1
\]

\[
(\tilde{a}_{21} \otimes \tilde{x}_1) \oplus (\tilde{a}_{22} \otimes \tilde{x}_2) \oplus \ldots \oplus (\tilde{a}_{2n} \otimes \tilde{x}_n) = \tilde{b}_2
\]

\[
(\tilde{a}_{m1} \otimes \tilde{x}_1) \oplus (\tilde{a}_{m2} \otimes \tilde{x}_2) \oplus \ldots \oplus (\tilde{a}_{mn} \otimes \tilde{x}_n) = \tilde{b}_n
\]

The matrix form of the above equations is

\[
\tilde{A} \otimes \tilde{x} = \tilde{b}
\]

Where the coefficient matrix \( \tilde{A} = (a_{ij}) \), \( 1 \leq i, j \leq n \) is a \( n \times n \) fuzzy matrix and \( \tilde{x}_j, \tilde{b}_j \in F(\mathbb{R}) \). This system is called a fully fuzzy linear system.

Result 2.11.

If \( A \) is an \( m \times n \) matrix with full column rank, then \( A \) can be factored as \( A = QR \) where \( Q \) is an \( m \times n \) matrix whose column vectors form an orthonormal basis for the column space \( A \) and \( R \) is a \( n \times n \) invertible upper triangular matrix.

3. DUAL FULLY FUZZY LINEAR SYSTEM

In this paper we are going to find a solution of dual fully fuzzy linear system

\[
\tilde{A}_1 \otimes \tilde{x} = \tilde{A}_2 \otimes \tilde{x} = \tilde{b}
\]

Where \( \tilde{A}_1 = (A_1, M_1, N_1) \), \( \tilde{A}_2 = (A_2, M_2, N_2) \), \( \tilde{b} = (b, g, h) \) and \( \tilde{x} = (x, y, z) \geq 0 \)

\[
(A_1, M_1, N_1) \otimes (x, y, z) = (A_2, M_2, N_2) \otimes (b, g, h)
\]

Using 2.7 (iii) we get

\[
(\tilde{a}_{11} x, \tilde{a}_{12} y + M_1 x, \tilde{a}_{13} z + N_1 x) = (\tilde{a}_{21} x, \tilde{a}_{22} y + M_2 x, \tilde{a}_{23} z + N_2 x) \otimes (b, g, h)
\]

Using 2.7 (ii) we get

\[
(\tilde{a}_{11} x, \tilde{a}_{12} y + M_1 x, \tilde{a}_{13} z + N_1 x) = (\tilde{a}_{21} x + b, \tilde{a}_{22} y + M_2 x + g, \tilde{a}_{23} z + N_2 x + h)
\]

Using 2.4 we get

\[
\tilde{A}_1 x = \tilde{A}_2 x + b
\]

\[
\Rightarrow (\tilde{A}_1 - \tilde{A}_2) x = b
\]

\[
\tilde{A}_1 y + M_1 x = \tilde{A}_2 y + M_2 x + g
\]

\[
\Rightarrow (\tilde{A}_1 - \tilde{A}_2) y = g - (M_1 - M_2) x
\]

\[
\tilde{A}_1 z + N_1 x = \tilde{A}_2 z + N_2 x + h
\]

\[
\Rightarrow (\tilde{A}_1 - \tilde{A}_2) z = h - (N_1 - N_2) x
\]

Let us take \( \tilde{A}_1 - \tilde{A}_2 = A \)

\[
(M_1 - M_2) = M, (N_1 - N_2) = N
\]

The above equations becomes
\[ Ax = b \]
\[ Ay = g - Mx \quad (3.2) \]
\[ Az = h - Nx \]

4. DUAL FULLY FUZZY LINEAR SYSTEM

4.1. Simplification of QR decomposition

Let \( (A_1, M_1, N_1) = (Q_1, 0, Q_3) \otimes (R_1, R_2, 0) \)

Using 2.7 (iii) we have

\[
\begin{align*}
(A_1, M_1, N_1) & = (Q_1 R_1, Q_1 R_2, Q_2 R_2) \\
A_1 & = Q_1 R_1 \quad \Rightarrow \quad R_1 = Q_1^{-1} A_1 \\
M_1 & = Q_1 R_2 \quad \Rightarrow \quad R_2 = Q_1^T M_1 \\
N_1 & = Q_2 R_2 \quad \Rightarrow \quad N_1 = R_1^{-1} N_1
\end{align*}
\]

and

\[](A_2, M_2, N_2) = (Q_1^*, 0, Q_3^*) \otimes (R_1^*, R_2^*, 0)

using 2.7(iii) we have

\[
\begin{align*}
(A_2, M_2, N_2) & = (Q_1^* R_1^*, Q_1^* R_2^*, Q_2^* R_2^*) \\
A_2 & = Q_1^* R_1^* \quad \Rightarrow \quad R_1^* = (Q_1^*)^{-1} A_2 \\
M_2 & = Q_1^* R_2^* \quad \Rightarrow \quad R_2^* = (Q_1^*)^{-1} M_2 \\
N_2 & = Q_2^* R_2^* \quad \Rightarrow \quad N_2 = N_1 \quad \text{(where matrices } Q_1 \text{ and } Q_1^* \text{ are orthonormal crisp matrices and matrices } R_1 \text{ and } R_1^* \text{ are upper triangular crisp matrices.)}
\end{align*}
\]

For solving dual fully fuzzy linear system (3.1) with this method.

Consider \( \tilde{A}_1 \otimes \tilde{x} = \tilde{A}_2 \otimes \tilde{x} \otimes \tilde{b} \)

\[
\begin{align*}
(A_1, M_1, N_1) \otimes (x, y, z) & = (A_2, M_2, N_2) \otimes (x, y, z) \\
& \otimes (b, g, h)
\end{align*}
\]

Using 2.7 (iii) we get

\[
\begin{align*}
(Q_1, 0, Q_3) \otimes (R_1, R_2, 0) \otimes (x, y, z) & = (Q_1^*, 0, Q_3^*) \otimes (R_1^*, R_2^*, 0) \otimes (x, y, z) \otimes (b, g, h)
\end{align*}
\]

Result 4.3.

Let \( \tilde{A} = (A, M, N) = (Q_1, 0, Q_3) \otimes (R_1, R_2, 0) \)

Using 2.7 (iii) we have

\[
\begin{align*}
(A, M, N) & = (Q_1 R_1, Q_1 R_2, Q_3 R_3) \\
A & = Q_1 R_1 \quad \Rightarrow \quad R_1 = Q_1^{-1} A \\
M & = Q_1 R_2 \quad \Rightarrow \quad R_2 = Q_1^T M \\
N & = Q_3 R_3 \quad \Rightarrow \quad N = Q_3 R_3
\end{align*}
\]

Consider \( \tilde{A} x = \tilde{b} \)

\[
\begin{align*}
(A_1, M_1, N_1) \otimes (x, y, z) & = \tilde{b} \\
(Q_1 R_1 x + Q_1 R_2 y + Q_1 R_3 z) & = \tilde{b} \\
Q_1 R_1 x + Q_1 R_2 y & = g \Rightarrow y = R_1^{-1} Q_1^T [g - M x] \\
Q_3 R_1 x + Q_1 R_2 z & = h \Rightarrow z = R_1^{-1} Q_1^T [h - N x]
\end{align*}
\]

5. NUMERICAL EXAMPLE
5.1. Solve the following dual fully fuzzy linear system Using QR Decomposition method

\[
(6,5,6) \oplus (x_1, y_1, z_1) \oplus (9,2,8) \oplus (x_2, y_2, z_2) = (2,2,1) \oplus (x_1, y_1, z_1) \oplus (3,0,1) \oplus (x_2, y_2, z_2) \oplus (24,23,33)
\]

\[
(6,4,9) \oplus (x_1, y_1, z_1) \oplus (8,5,9) \oplus (x_2, y_2, z_2) = (1,1,3) \oplus (x_1, y_1, z_1) \oplus (3,3,2) \oplus (x_2, y_2, z_2) \oplus (25,23,37)
\]

Solution:

\[
A_1 = \begin{bmatrix} 6 & 9 \\ 6 & 8 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 5 & 2 \\ 4 & 5 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}
\]

\[
N_1 = \begin{bmatrix} 6 & 9 \\ 9 & 8 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}, \quad g = \begin{bmatrix} 23 \\ 23 \end{bmatrix}, \quad h = \begin{bmatrix} 33 \\ 37 \end{bmatrix}
\]

\[
A_1 = Q_1 R_1 = \begin{bmatrix} \frac{6}{\sqrt{72}} & \frac{1}{\sqrt{72}} \\ \frac{6}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{72} & 102 \\ \sqrt{72} & -1 \end{bmatrix}
\]

\[
A_2 = Q_2^T R_2 = \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{5} & 9 \\ \sqrt{5} & -1 \end{bmatrix}
\]

\[
R_2 = Q_2^T M_1 = \begin{bmatrix} \frac{54}{\sqrt{5}} & 42 \\ \frac{1}{\sqrt{5}} & -3 \end{bmatrix}
\]

\[
Q_3 = N_1 R_1^{-1} = \frac{1}{6} \begin{bmatrix} 6 & -36 \\ \frac{6}{\sqrt{72}} & \sqrt{72} \end{bmatrix}
\]

\[
R_2^* = (Q_2^*)^T M_2 = \begin{bmatrix} \sqrt{5} & 2 \\ 0 & \frac{2}{\sqrt{5}} \end{bmatrix}
\]

\[
Q_5 = N_2 (R_2^*)^{-1} = \frac{1}{3} \begin{bmatrix} 3 & 0 \\ \sqrt{5} & \frac{2}{\sqrt{5}} \end{bmatrix}
\]

\[
x = (Q_1 R_1 - Q_1^* R_2^*)^{-1} b = \begin{bmatrix} 3 \\ 2 \end{bmatrix}
\]

\[
y = (Q_2 R_2 - Q_2^* R_5^*)^{-1} \left[ g - (Q_1 R_2 - Q_1^* R_2^*) x \right] = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

\[
z = (Q_1 R_1 - Q_1^* R_2^*)^{-1} \left[ h - (Q_3 R_3 - Q_3^* R_3^*) x \right] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

\[
\hat{x}_1 = (3,1,1), \quad \hat{x}_2 = (2,1,0)
\]

\[ z = R_1^{-1}Q_1^T[h - N \bar{x}] = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

\[ \bar{x}_1 = (3,1,1), \bar{x}_2 = (2,1,0) \]

5.3 Verification for Numerical example 5.1.

Using 2.7(i) and 2.7(iii)

\[
\begin{align*}
(6,5,6) \otimes (3,1,1) & \oplus (9,2,8) \otimes (2,1,0) = (36,34,40) \\
(2,2,1) \otimes (3,1,1) & \oplus (3,0,1) \otimes (2,1,0) \oplus (24,23,33)
\end{align*}
\]

= (36,34,40)

\[
\begin{align*}
(6,4,9) \otimes (3,1,1) & \oplus (8,5,9) \otimes (2,1,0) = (34,36,51) \\
(1,1,3) \otimes (3,1,1) & \oplus (3,3,2) \otimes (2,1,0) \oplus (25,23,37)
\end{align*}
\]

= (34,36,51)

5.4 Verification for Numerical example 5.2.

Using 2.7(i) and 2.7(iii)

\[
\begin{align*}
(6,5,6) \otimes (3,1,1) & \oplus (9,2,8) \otimes (2,1,0) = (36,34,40) \\
(2,2,1) \otimes (3,1,1) & \oplus (3,0,1) \otimes (2,1,0) \oplus (24,23,33)
\end{align*}
\]

= (36,34,40)

\[
\begin{align*}
(6,4,9) \otimes (3,1,1) & \oplus (8,5,9) \otimes (2,1,0) = (34,36,51) \\
(1,1,3) \otimes (3,1,1) & \oplus (3,3,2) \otimes (2,1,0) \oplus (25,23,37)
\end{align*}
\]

= (34,36,51)

\[
(2,4,6) \otimes (3,1,1) \oplus (7,8,9) \otimes (2,1,0) = (20,37,38)
\]

(1,2,3) \oplus (3,1,1) \oplus (2,4,6) \oplus (2,1,0) \oplus (13,20,16)

= (20,37,38)

6. CONCLUSION

In this paper, the solution of Dual fully fuzzy linear system of the form \( \bar{A}_1 \otimes \bar{x} = \bar{A}_2 \otimes \bar{x} \oplus \bar{b} \) whose coefficients are triangular fuzzy numbers is obtained by QR decomposition method. This method is useful when the system is square as well as rectangular.

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